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Dualities for a Class of Finite Range Probabilistic Cellular Automata in One Dimension

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In this paper we study dualities for a class of one-dimensional probabilistic cellular automata with finite range interactions by using a sequence of extended cellular automata.

KEY WORDS: Duality; cellular automata; finite range.

1. INTRODUCTION

The models considered in this paper are probabilistic cellular automata with finite range interactions on **Z**, which is the set of integers. Before we study the finite range case, for better understanding of readers, we review and explain the pairwise interaction case based on the paper written by Katori *et al.*[(1)] Our main result (Theorem 2) is an extention of their result (Theorem 1) for a class of models with finite range interactions which is called Class SB (N) in the present paper.

The one-dimensional probabilistic cellular automata with pairwise interactions was introduced by Domany and Kinzel[(2)] and Kinzel.[(3)] So we call this class the *Domany–Kinzel model*. This model is a two parameter family of discrete time Markov processes whose states are subsets of **Z**. Let $\xi_n^A \subset \mathbf{Z}$ be the state of the process with parameters $(p_1, p_2) \in [0, 1]^2$ at time n which starts from $A \subset 2\mathbf{Z}$. Its evolution satisfies the following.

(i) $P(x \in \xi_{n+1}^A \mid \xi_n^A) = f(|\xi_n^A \cap \{x-1, x+1\}|)$,

(ii) given ξ_n^A, the events $\{x \in \xi_{n+1}^A\}$ are independent, where

$$f(0) = 0, \qquad f(1) = p_1, \qquad \text{and} \qquad f(2) = p_2$$

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where $|A|$ is the number of elements in A. If we write $\xi(x, n) = 1$ for $x \in \xi_n^A$ and $\xi(x, n) = 0$ otherwise, each realization of the process is identified with a configuration $\xi \in \{0, 1\}^S$ with $S = \{s = (x, n) \in \mathbf{Z} \times \mathbf{Z}_+ : x + n = \text{even}\}$, where $\mathbf{Z}_+ = \{0, 1, 2,...\}$.

As special cases the Domany–Kinzel model is equivalent to the oriented bond percolation model ($p_1 = p$, $p_2 = 2p - p^2$) and the oriented site percolation model ($p_1 = p_2 = p$) on a square lattice. The two-dimensional mixed site-bond oriented percolation model with probabilities p_s of a site being open and p_b of a bond being open corresponds to the case of $p_1 = p_s p_b$ and $p_2 = p_s[1-(1-p_b)^2]$. The model with $(p_1, p_2) = (1, 0)$ becomes Wolfram's[4, 5] rule 90. For more detailed information, see pp. 90–98 in Durrett.[6]

A key observation in this paper is that any two parameters $p_1, p_2 \in [0, 1]$ can be expressed as

$$p_1 = p_s p_b, \qquad p_2 = p_s[1-(1-p_b)^2] \tag{1.1}$$

where $p_s, p_b \in \mathbf{R}$ (the set of real numbers). From these we have

$$p_s = \frac{p_1^2}{2p_1 - p_2}, \qquad p_b = 2 - \frac{p_2}{p_1} \tag{1.2}$$

where $2p_1 \neq p_2$ and $p_1 \neq 0$. The expression (1.1) is equivalent to that of the mixed site-bond oriented percolation for $p_s, p_b \in [0, 1]$. However in the Domany–Kinzel model, $p_s, p_b \in [0, 1]$ is not assumed. In this paper (1.1) will be called *Expression SB*.

Next a new process η_n is introduced as follows. To do so we let

$$p_* = \max\{p_1, p_2\}$$

For simplicity, from now on we suppose that $2p_1 \neq p_2$ and $p_1 \neq 0$, when we consider the Domany–Kinzel model. A new process defined below is called *p_*-DKdual*. We can see the thinning-relationship by coupling the Domany–Kinzel model and p_*-DKdual. We split both the Domany–Kinzel model and the p_*-DKdual into two phases, and we will allow the first phase to occur at times $n + (1/2)$ where n is an integer.

1. Let μ be the distribution of the p_*-DKdual at time 0.

2. At time $n = 1/2$, it undergoes a p_*-thinning. In general, for $p \in [0, 1]$, the p-thinning of a set $A \subset \mathbf{Z}$ is the random subset of A obtained by independently removing each element of A with probability $1 - p$.

3. Start the Domany–Kinzel model at time $n = 0$ with the same configuration as the p_*-DKmodel at time $n = 1/2$.

4. Couple the processes together until time $n_0-(1/2)$ for the Domany–Kinzel, n_0 for the p_*-DKmodel. This can be done because the transitions for the Domany–Kinzel model are the same as those for the p_*-DKmodel lagged by time unit $1/2$.

5. Perform a p_*-thinning for the Domany–Kinzel model at time n_0.

The distribution of the Domany–Kinzel model started and ended as a p_*-thinning of the p_*-DK dual.

Here we consider dualities for the Domany–Kinzel model ξ_n^A and the p_*-DKdual η_n^A starting from A. In general, duality is a very useful technique in the study of probabilistic cellular automata. Because problems in uncountable state space (typically configurations of zeros and ones live in **Z**) can be reformulated as problems in countable state space (typically finite subsets of **Z**). See Chapter 5 of Durrett[6] for some applications of duality. The following theorem was shown by Katori *et al.*[1] Remark that they assumed $p_1 \geqslant p_2$ in their paper, so $p_* = p_1$.

Theorem 1. For any A, B with $|A| < \infty$ or $|B| < \infty$, we have

(1) $E(\alpha^{|\xi_n^A \cap B|}) = E(\alpha^{|\xi_n^B \cap A|})$ for any $n \geqslant 0$, if $\alpha = 1-(2p_1-p_2)/p_1^2$,

(2) $E(\beta^{|\xi_n^A \cap B|}) = E(\beta^{|\eta_n^B \cap A|})$ for any $n \geqslant 0$, if $\beta = 1-(2p_1-p_2)\, p_*/p_1^2$,

(3) $E(\gamma^{|\eta_n^A \cap B|}) = E(\gamma^{|\eta_n^B \cap A|})$ for any $n \geqslant 0$, if $\gamma = 1-(2p_1-p_2)\, p_*^2/p_1^2$.

We should remark that if we use p_s in Eq. (1.2) then

$$\alpha = 1-\frac{1}{p_s}, \qquad \beta = 1-\frac{p_*}{p_s}, \qquad \gamma = 1-\frac{p_*^2}{p_s} \tag{1.3}$$

When $p_2 = 1-(1-p_1)^2$ (oriented bond percolation case) in part (1), we have $\alpha = 0$. So in this case, we obtain

$$P(\xi_n^A \cap B \neq \emptyset) = P(\xi_n^B \cap A \neq \emptyset)$$

for any $n \geqslant 0$. This result is known as coalescing self-duality for oriented bond percolation. Moreover, continuous time version of part (1) in this theorem has been studied by Sudbury and Lloyd,[7,8] Bandt,[9] and Sudbury.[10]

From now on we turn to a general finite range case. To describe the dynamics of probabilistic cellular automaton with finite range interactions, we introduce an interaction neighborhood $\mathcal{N} = \{-L, -(L-1),..., L-1, L\}$ and a transition function F: $\{0, 1\}^{\mathcal{N}} \to [0, 1]$. In this setting, its evolution is specified by

(i) $P(x \in \xi_{n+1}^A \mid \xi_n^A) = F(\xi_n^A(x-L),..., \xi_n^A(x+L))$,

(ii) given ξ_n^A, the events $\{x \in \xi_{n+1}^A\}$ are independent.

Here we let $F(i_1, i_2,..., i_N) = p_{i_1 i_2 \cdots i_N} \in [0, 1]$ for any $i_k \in \{0, 1\}$ $(k = 1, 2,..., N)$ with $N = 2L+1$. We call this model *PCA* for short.

As in the case of the Domany–Kinzel model, we introduce a new process p_*-PCA η_n with respect to the PCA ξ_n, where $p_* = \max\{p_{i_1 i_2 \cdots i_N}: i_1, i_2,..., i_N \in \{0, 1\}\}$. Furthermore, we impose the following Expression SB, which is a finite range version of the Domany–Kinzel model (see Eq. (1.1)), on transition probabilities $p_{i_1 i_2 \cdots i_N}$ of the PCA:

$$
\begin{aligned}
p_{0000...0000} &= 0,\\
p_{0000...0001} &= p_{1000...0000} = p_s p_L,\\
p_{0000...0010} &= p_{0100...0000} = p_s p_{L-1},\\
p_{0000...0100} &= p_{0010...0000} = p_s p_{L-2},\\
p_{0000...0011} &= p_{1100...0000} = p_s[1-(1-p_{L-1})(1-p_L)],\\
p_{0000...0101} &= p_{1010...0000} = p_s[1-(1-p_{L-2})(1-p_L)],\\
p_{0000...0111} &= p_{1110...0000} = p_s[1-(1-p_{L-2})(1-p_{L-1})(1-p_L)],\\
&\vdots\\
p_{1111...1111} &= p_s\left[1-(1-p_0)\prod_{k=1}^{L}(1-p_k)^2\right]
\end{aligned}
$$

where $p_s \in \mathbf{R} \setminus \{0\}$ and $p_k \in \mathbf{R}$ $(k = 0, 1,..., L)$ satisfy $p_{i_1 i_2 \cdots i_N} \geqslant 0$ for any $i_j \in \{0, 1\}$ $(j = 1, 2,..., N)$. This expression is necessary for the PCA to be self-dual. We call a class of models whose transition probabilities given by the above Expression SB *Class SB (N)*. The Domany–Kinzel model is equivalent to Class SB (3) with $p_0 = 0$, so it is essentially two-neighbor system. Then we obtain the next our main result for finite range case:

Theorem 2. We consider Class SB(N) with transition probabilities $p_{i_1 i_2 \cdots i_N}$. Then, for any A, B with $|A| < \infty$ or $|B| < \infty$, we have

(1) $E(\alpha^{|\xi_n^A \cap B|}) = E(\alpha^{|\xi_n^B \cap A|})$ for any $n \geqslant 0$, if $\alpha = 1 - 1/p_s$,

(2) $E(\beta^{|\xi_n^A \cap B|}) = E(\beta^{|\eta_n^B \cap A|})$ for any $n \geqslant 0$, if $\beta = 1 - p_*/p_s$,

(3) $E(\gamma^{|\eta_n^A \cap B|}) = E(\gamma^{|\eta_n^B \cap A|})$ for any $n \geqslant 0$, if $\gamma = 1 - p_*^2/p_s$,

where $p_* = \max\{p_{i_1 i_2 \cdots i_N}: i_1, i_2,..., i_N \in \{0, 1\}\}$.

Noting the expression (1.3), it is clear that Theorem 2 is an extension of Theorem 1.

2. PROOF OF THEOREM 2

In this section, we prove Theorem 2. However the proof is the essentially same as the proof of Theorem 1 for the Domany–Kinzel model. So we show the theorem in the case of the Domany–Kinzel model, that is, Theorem 1. As we mentioned just before Theorem 1, Theorem 1 was proved by Katori *et al.*[1] However their proofs of (2) and (3) can not be extended to our finite range case. On the other hand, our proofs here are easy to extend to a general case, moreover they clarify a reason why duality parameters α, β and γ in this theorem satisfy a simple relation $p_*^2(1-\alpha)=p_*(1-\beta)=1-\gamma$ by introducing a sequence of extended Domany–Kinzel models $\xi_n^{(k),A}$ for $k=0,1,2,...$, where $k=0$ is equivalent to the original Domany–Kinzel model. We use $\xi_n^{(k-1),A}$ in the proof of part (k) for each $k=1,2,3$. Furthermore, an interesting thing is that parameters $(p_1^{(k)}, p_2^{(k)})$ $(=(p_1/p_*^k, p_2/p_*^k))$ of $\xi_n^{(k),A}$ for $k\geqslant 2$ do not belong to $[0,1]^2$ unless $p_*=1$. However the fact does not matter in our proof of part (3), since it is technically useful for our computation to consider such an artificial process.

To introduce the class of models $\xi_n^{(k),A}$ $(k=0,1,2,...)$ is an essential new part of this paper. For the convenience of readers, we present one of their three proofs of part (1) in our setting corresponding to our proofs of parts (2) and (3), since Katori *et al.*[1] gave three different proofs as for part (1).

Proof of (1). By the Markov property of ξ_n, it is enough to show

$$E(\alpha^{|\xi_1^A\cap B|})=E(\alpha^{|\xi_1^B\cap A|})$$

Let $\partial A=(A-1)\,\triangle\,(A+1)$ and $\hat{A}=(A-1)\cap(A+1)$ where $A+m=\{x+m: x\in A\}$ and $A\,\triangle\,B=(A\setminus B)\cup(B\setminus A)$. First we see that

$$\begin{aligned}E(\alpha^{|\xi_1^A\cap B|})&=\sum_{l=0}^{|\partial A\cap B|}\binom{|\partial A\cap B|}{l}\alpha^l p_1^l(1-p_1)^{|\partial A\cap B|-l}\\&\quad\times\sum_{k=0}^{|\hat{A}\cap B|}\binom{|\hat{A}\cap B|}{k}\alpha^k p_2^k(1-p_2)^{|\hat{A}\cap B|-k}\\&=(\alpha p_1+1-p_1)^{|\partial A\cap B|}(\alpha p_2+1-p_2)^{|\hat{A}\cap B|}\end{aligned}$$

Similarly we have

$$E(\alpha^{|\xi_1^B \cap A|}) = (\alpha p_1 + 1 - p_1)^{|\partial B \cap A|} \, (\alpha p_2 + 1 - p_2)^{|\hat{B} \cap A|}$$

where $\partial B = (B-1) \triangle (B+1)$ and $\hat{B} = (B-1) \cap (B+1)$. Let $\mathbf{a} = (a_i : i \in \mathbf{Z})$ and $\mathbf{b} = (b_i : i \in \mathbf{Z})$ with $a_i, b_i \in \{0, 1\}$ for any i, where, if $2i \in A$ (resp. $\notin A$), then $a_i = 1$ (resp. $= 0$) and if $2i+1 \in B$ (resp. $\notin B$), then $b_i = 1$ (resp. $= 0$). A little thought reveals

$$|\partial A \cap B| = \sum_i b_i (a_i + a_{i+1} - 2 a_i a_{i+1}),$$

$$|\partial B \cap A| = \sum_i a_{i+1} (b_i + b_{i+1} - 2 b_i b_{i+1}),$$

$$|\hat{A} \cap B| = \sum_i b_i a_i a_{i+1}, \qquad |\hat{B} \cap A| = \sum_i a_{i+1} b_i b_{i+1}$$

Therefore if we let $M = M(\mathbf{a}, \mathbf{b}) = \sum_i (a_{i+1} b_i b_{i+1} - a_i a_{i+1} b_i)$, then we have

$$|\partial A \cap B| = |\partial B \cap A| + 2M, \qquad |\hat{A} \cap B| = |\hat{B} \cap A| - M \tag{2.1}$$

Here we assume that $\alpha p_2 + 1 - p_2 = (\alpha p_1 + 1 - p_1)^2$, i.e., $\alpha = 1 - (2p_1 - p_2)/p_1^2$. Combining (2.1) with the above assumption gives

$$\begin{aligned} E(\alpha^{|\xi_1^A \cap B|}) &= (\alpha p_1 + 1 - p_1)^{|\partial A \cap B|} \, (\alpha p_1 + 1 - p_1)^{2\,|\hat{A} \cap B|} \\ &= (\alpha p_1 + 1 - p_1)^{|\partial A \cap B| + 2\,|\hat{A} \cap B|} \\ &= (\alpha p_1 + 1 - p_1)^{|\partial B \cap A| + 2\,|\hat{B} \cap A|} \\ &= E(\alpha^{|\xi_1^B \cap A|}) \end{aligned}$$

So we obtain the desired conclusion.

Proof of (2). First we consider an extended Domany–Kinzel model $\xi_n^{(k), A}$ starting from A with parameters $(p_1^{(k)}, p_2^{(k)}) = (p_1/p_*^k, p_2/p_*^k)$ for any $k \in \{0, 1, \ldots\}$. By using part (1), we have

$$E((\alpha^{(k)})^{|\xi_1^{(k), A} \cap B|}) = E((\alpha^{(k)})^{|\xi_1^{(k), B} \cap A|}) \tag{2.2}$$

where

$$\alpha^{(k)} = 1 - \frac{2p_1^{(k)} - p_2^{(k)}}{(p_1^{(k)})^2} = 1 - \frac{2p_1 - p_2}{p_1^2} \times p_*^k = 1 - \frac{p_*^k}{p_s} \tag{2.3}$$

since $p_s = p_1^2/(2p_1 - p_2)$. In the general finite range case, we consider extended PCA with transition probabilities $p_{i_1 i_2 \cdots i_N}/p_*^k$ in the proof of part $(k+1)$ for $k = 1, 2$.

From the above Eqs. (2.2) and (2.3) with $k = 1$, we see that

$$\begin{aligned}
E((\alpha^{(1)})^{|\xi_1^A \cap B|}) &= E((\alpha^{(1)})^{|\xi_{1/2}^{(1),A} \cap B_{p_*}|}) \\
&= \sum_{D \subset B} p_*^{|D|}(1-p_*)^{|B|-|D|}\, E((\alpha^{(1)})^{|\xi_{1/2}^{(1),A} \cap D|}) \\
&= \sum_{D \subset B} p_*^{|D|}(1-p_*)^{|B|-|D|}\, E((\alpha^{(1)})^{|\xi_{1/2}^{(1),D} \cap A|}) \\
&= E((\alpha^{(1)})^{|\xi_{1/2}^{(1),B_{p_*}} \cap A|}) \\
&= E((\alpha^{(1)})^{|\eta_1^B \cap A|})
\end{aligned}$$

where B_p is a p-thinning for B, that is, if $x \in B$, then $x \in B_p$ with probability p and $x \notin B_p$ with probability $1-p$. Noting that

$$\alpha^{(1)} = \beta = 1 - \frac{(2p_1 - p_2)\, p_*}{p_1^2} = 1 - \frac{p_*}{p_s}$$

the proof is completed.

Proof of (3). As in the case of part (2), by using Eqs. (2.2) and (2.3) with $k = 2$, we have

$$\begin{aligned}
E((\alpha^{(2)})^{|\eta_1^A \cap B|}) &= E((\alpha^{(2)})^{|\xi_{1/2}^{(1),A_{p_*}} \cap B|}) \\
&= E((\alpha^{(2)})^{|\xi_{1/4}^{(2),A_{p_*}} \cap B_{p_*}|}) \\
&= \sum_{D \subset B} p_*^{|D|}(1-p_*)^{|B|-|D|}\, E((\alpha^{(1)})^{|\xi_{1/2}^{(1),A} \cap D|}) \\
&= \sum_{D \subset A} \sum_{F \subset B} p_*^{|D|}(1-p_*)^{|A|-|D|}\, p_*^{|F|}(1-p_*)^{|B|-|F|}\, E((\alpha^{(2)})^{|\xi_{1/4}^{(2),D} \cap F|}) \\
&= \sum_{D \subset A} \sum_{F \subset B} p_*^{|D|}(1-p_*)^{|A|-|D|}\, p_*^{|F|}(1-p_*)^{|B|-|F|}\, E((\alpha^{(2)})^{|\xi_{1/4}^{(2),F} \cap D|}) \\
&= E((\alpha^{(2)})^{|\eta_1^B \cap A|})
\end{aligned}$$

Then the proof is completed, since

$$\alpha^{(2)} = \gamma = 1 - \frac{(2p_1 - p_2)\, p_*^2}{p_1^2} = 1 - \frac{p_*^2}{p_s}$$

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